EFFECTIVE CONDUCTIVITY OF A BODY WITH A LARGE NUMBER OF CRACKS TAKING INTO ACCOUNT THEIR CAPACITANCE AND THE ACTION OF MECHANICAL LOADS

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The problem of cracks acting as electrical capacitors and subjected to mechanical loads is considered.

The problem of a crack subjected to a mechanical load in an electric field acting as a capacitance is considered, and for a body with a large number of noninteracting cracks a relation is obtained between the effective current density and the electric field strength. Note that in view of the well-known mathematical analogy this problem can also be considered as the problem of heat conduction (the current density corresponds to the heat flux, and the electric field strength corresponds to the temperature gradient with opposite sign). For the medium inside the crack there is a nonquasistationary heat-conduction law: in the expression for the heat flow there is a term proportional to the rate of variation of the temperature gradient.

For fairly rapid changes in the currents, due either to the sources producing them or due to changes in the opening of the cracks because of mechanical loads, or both together, cracks with a high resistance of the medium between them, operating as electric capacitors, begin to pass current. This effect is considered below when the rate of change of the electric fields, while being sufficient for the effect to appear, at the same time is not too large, so that the electric-field distribution around the crack can be assumed at each instant of time to satisfy the steady-state equations (the skin effect can be neglected for distances of the order of the size of a crack).

1. If the crack has an elliptical shape in plan, and is opened by a uniform field of stresses, and also, possibly, by the action of a uniformly distributed internal pressure, its shape is described by an ellipsoid. The problem of an ellipsoid in a conducting medium possessing properties different from it and situated in a uniform electric field has been solved accurately; the electric field inside the ellipsoid turns out to be uniform.

Below, for simplicity, we will consider a disk-shaped crack of radius a. In the loaded state its change in thickness is described by an ellipsoid of rotation with polar semiaxis c and radius of the equator a.

Assuming the material to be isotropic, and denoting by σ the field component of the mechanical stresses far from the crack normal to the crack, and by p the pressure in the crack, we have for its normal elastic semiopening w [1]

$$w=\frac{4\left(1-\nu_0^2\right)}{\pi E_0}(\sigma+p)\sqrt{a^2-r^2}, \tag{1}$$

where ν_0 is Poisson's ratio of the material. For a crack with a negligibly small initial thickness, c = w. Henceforth, we will consider the more general case when the initial thickness of the crack is also described by an ellipsoid of rotation with minor semiaxis c_0 and radius a. Then

$$c=c_0+w. \tag{2}$$

We will assume that the electric fields and currents are not so strong as to produce mechanical stresses, which could affect the opening of the crack.

The electrical conductivity of the material will be denoted by λ_0. Using the well-known analogy between problems of electrical conductions, we can write for the field inside

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the ellipsoid relations corresponding to the electrostatic problem [2], replacing the electric induction vector by the current-density vector, and the permittivity of the material by its electrical conductivity.

We thereby obtain for the field components perpendicular to the crack

$$(1-n_\perp)\lambda_0 G_\perp^{(i)} + n_\perp j_\perp^{(i)} = \lambda_0 G_\perp = j_\perp . \tag{3}$$

Here we denote by G and j the electric field strength and the current density far from the crack, the superscript (i) denotes the corresponding quantity inside the crack, and $n_\perp = \beta^{-3}(1+\beta^2)(\beta - \operatorname{arctg}\beta)$, $\beta = \sqrt{(a/c)^2 - 1}$. A similar relation holds for the field components in the plane of the crack (which we will denote by the index $\|$) with $n_\perp$ replaced by $n_\| = 1/2(1-n)$. Taking (1) and (2) into account and the fact that $(c/a) \ll 1$, we have

$$n_\perp = 1 - s,\quad n_\| = \frac{1}{2}s,\quad s = \frac{\pi c_0}{2a} + \frac{\sigma + p}{2E_0}(1-\nu_0^2). \tag{4}$$

These relations, because of the uniformity of the field inside the ellipsoid, hold for any relation between $j_\perp^{(i)}$ and $G_\perp^{(i)}$, and $j_\|^{(i)}$ and $G_\|^{(i)}$.

The medium in the crack will be assumed to be slightly conducting so that it can simultaneously be represented by a permittivity ε_c and an electrical conductivity λ_c. The current density in this medium is made up of the displacement current density and the conduction current density

$$\mathbf{j}^{(i)} = \varepsilon_c \frac{d\mathbf{G}^{(i)}}{dt} + \lambda_c G^{(i)}. \tag{5}$$

From (3)-(5), neglecting s compared with unity, we obtain

$$\frac{dG_\perp^{(i)}}{dt} + \frac{1}{\tau_s} G_\perp^{(i)} = \frac{1}{\tau} G_\perp . \tag{6}$$

Here

$$\gamma = \lambda_c \lambda_0^{-1},\quad \tau = \varepsilon_c \lambda_0^{-1},\quad \tau_s = \tau(s+\gamma)^{-1}.$$

Similarly, substituting into (3) $n_\|$ from (4) instead of $n_\perp$, and using (2), we obtain with the same accuracy

$$\frac{dG_\|^{(i)}}{dt} + \frac{2+\gamma s}{s\tau} G_\|^{(i)} = \frac{2}{s\tau} G_\| . \tag{7}$$

It can be seen from (6) that the characteristic time of the variation of $G_\perp^{(i)}$ is of the order of τ_s. When considering processes with a characteristic time of this order the derivative in (7) can be omitted: its ratio to the term following it is of the order of $s(s+\gamma)(2+\gamma s)^{-1}$. Consequently, $G_\|^{(i)} = (1 + {}^1/_2\gamma s)$. The most interesting case is when the opening of the crack has an effect on τ_s, i.e., when γ is of the order τ_s or less.

This case is also considered below. Solving Eq. (6), we obtain

$$G_\perp^{(i)} = G_{\perp 0}^{(i)} \exp\left(-\int_{t_0}^{t} \frac{dt'}{\tau_s(t')}\right) + \frac{1}{\tau}\int_{t_0}^{t} G_\perp(t') \exp\left(-\int_{t'}^{t} \frac{dt''}{\tau_s(t'')}\right) dt'. \tag{8}$$

Here $G_{\perp 0}$ is the value of the normal component of the electric field in the crack when $t = t_0$. If there is no electric field at the initial instant, this quantity is equal to zero. If there is a steady state at the initial instant (in this case $s = s_0$ and $G_\perp = G_{\perp 0}$), then, equating the derivative in (6) to zero, we obtain $G_{\perp 0}^{(i)} = G(s_0 + \gamma)^{-1}$. Hence we see that inside the crack the normal component of the electric field is far greater than far away. This, as can be seen from (8), also occurs in the nonsteady state mode when the variations of the electric field occur in a time of the order of τ_s: the second term is of the order of $G_\perp \tau_s/\tau$, i.e., $G_\perp(s_* + \gamma)^{-1}$, where s_* is on the order of s.

On the other hand, the tangential component of the field, due to the smallness of $s\gamma$, hardly changes when crossing into the crack, while the tangential component of the current density is not more than a $(s_* + \gamma)$ part of the normal component. We will now clarify the conditions under which the expressions obtained are applicable.

In order that the displacement currents in the body can be neglected compared with the conduction currents, as was done above, it is necessary for the changes in the electric field occurring during the characteristic time τ_s should not be very rapid. If the body is not a good conductor and can be represented by a constant permittivity ε_0 in addition to λ_0, this requirement means that we must have $\tau_s \ll \varepsilon_0\lambda_0^{-1}$ or, taking (7) into account

$$s_* + \gamma \ll \varepsilon_c\varepsilon_0^{-1}. \quad (9)$$

For good conductors ε_0 here means the corresponding index quantity. Under normal conditions (9) can always be assumed to be satisfied due to the smallness of $s_* + \gamma$.

In order that the electric field around the crack can be assumed at each instant of time to satisfy the steady-state equations, the characteristic time τ_s must be far greater than the buildup time of the electric field around the crack, which is of the order of $a^2\lambda_0\mu$, where μ is the magnetic permeability of the body. This leads to the condition

$$a \ll [\varepsilon_c/\mu(s_* + \gamma)]^{1/2}\lambda_0^{-1}, \quad (10)$$

i.e., for a given crack size the approximation considered is better the higher the resistivity of the body λ_0^{-1} (note that the quantity on the right in (10) corresponds to the skin depth in the body at a frequency of the order τ_s^{-1}).

The resistivity of materials varies over an extremely wide range so various cases are possible. We will confine ourselves to a single example. We will take $\lambda_0^{-1} = 10^6\ \Omega\cdot$m. These values are characteristic for a number of rocks and constructional materials. The values of ε_c and μ will be taken as approximately equal to their values for a vacuum, viz., $10^{-9}\ (36\pi)^{-1}$ sec$\cdot\Omega^{-1}\cdot$m^1 and $4\pi\cdot10^{-7}\ \Omega\cdot$sec$\cdot$m^{-1}. We will take $s_* + \gamma = 10^{-2}$–10^{-3} (the necessary smallness of γ is ensured for many versions of the filling of cracks, in particular, for fillings with gases and polymer materials). For the assumed values of the quantities we have $\tau_s \simeq 10^{-2}$–10^{-3} sec, and condition (10) is satisfied with a large margin for normal crack sizes.

In view of the fact that $G_\perp^{(i)} \gg G_\perp$, a situation is possible where electric breakdown occurs in the crack. Consideration of these phenomena is outside the scope of this article.

We also note that in the surrounding material when approaching the contour of the crack the field strength and current density formally become infinite. In fact, there is a small end region where they are very large. This fact can be important under fracture conditions at the end of the crack.

2. The relation between the effective current density and the electric field strength for a body containing a large number of cracks can be obtained by averaging the true values of these quantities over the volume (see, e.g., [2]), which we denote by angular brackets. The true values of the quantities will be denoted by primes, and quantities averaged over the volume will be written without primes. Without loss of generality, the averaging volume can be assumed to be unity. The cracks are assumed to be noninteracting (low concentration).

In Cartesian components (k = 1, 2, 3) we have

$$\langle j_k' - \lambda_0 G_k' \rangle = j_k - \lambda_0 G_k. \quad (11)$$

On the other hand, since the averaged expression is zero outside the crack

$$\langle j_k' - \lambda_0 G_k' \rangle = \frac{8}{3}\Sigma a^3 s\,(j_k^{(i)} - \lambda_0 G_k^{(i)}). \quad (12)$$

Here the superscript (i) denotes, as previously, the fields inside the cracks, and since they are uniform in the cracks, the integration over the volume of the crack can be replaced by multiplication by this volume, and summation is extended to all the cracks in unit volume. The terms in (12) are expressed in terms of G_k using the relations in Sec. 1. As a result, we obtain an expression for j_k in terms of G_k from (11) and (12).

Neglecting all terms on the order of s_* and higher on the right side of (12), only terms of the order of s_*^{-1} should remain in the round brackets. These terms, as was shown in Sec. 1, can only arise due to electric field components normal to the crack. Hence,

$$j_k^{(i)} - \lambda_0 G_k^{(i)} = n_k\,(j_\perp^{(i)} - \lambda_0 G_\perp^{(i)}). \quad (13)$$

Here n_k are the components of the vector of the unit normal to the crack which are expressed in terms of the spherical width ϑ and length ψ using the equations

$$n_1 = \sin\vartheta\cos\psi, \quad n_2 = \sin\vartheta\sin\psi, \quad n_3 = \cos\vartheta. \quad (14)$$

Below the set of angles ϑ, ψ will be denoted by Ω, while the element of the solid angle $\sin\vartheta\, d\vartheta\, d\psi$ will be denoted by $d\Omega$.

An expression for $j_\perp^{(i)} - \lambda_0 G_\perp^{(i)}$ is obtained from (8) and (5). Retaining only terms on the order of s_*^{-1} (or, which amounts to the same thing, $(s_* + \gamma)^{-1}$), we have

$$j_\perp^{(i)} - \lambda_0 G_\perp^{(i)} = -\lambda_0 (s+\gamma)^{-1} M, \tag{15}$$

$$M = \frac{n_l}{\tau_s} \int_0^{t-t_0} G_l(t-t') \exp\left(-\int_{t-t'}^{t} \frac{dt''}{\tau_s(t'')}\right) dt' + (s+\gamma) G_{\perp 0}^{(i)} \exp\left(-\int_{t_0}^{t} \frac{dt'}{\tau_s(t')}\right), \tag{16}$$

$$s = \frac{\pi c_0}{2a} + \frac{\sigma_{lm} n_l n_m + p}{2E_0}(1-\nu_0^2), \quad \tau_s = \frac{\varepsilon_c}{\lambda_0 (s+\gamma)}. \tag{17}$$

We have used here the expressions $G_\perp = G_l n_l$; $\sigma = \sigma_{lm} n_l n_m$; σ_{lm} are the components of the mechanical stress tensor; the recurrent indices mean summation from 1 to 3, and when changing to (16) the variable of integration t' is replaced by t − t'. Note that for a sufficiently large elastic opening of the cracks, when the first term in s can be neglected, the explicit dependence of τ_s on a and c_0 disappears.

When the transient begins from the initial steady state, $G_{\perp 0}^{(i)} = n_l G_l(0)(s_0 + \gamma)^{-1}$, and the second term in (16) is finite.

We will denote by Z the set of parameters$\{a, c_0, t_0, p_0, \varepsilon_c, \lambda_c\}$, where p_0 is the initial pressure in the crack; the product of the differentials of these parameters will be denoted by dZ. These parameters, like Ω, may vary from crack to crack. The inclusion in Z of the quantity t_0 may be important if the cracks occur under a load. We will also assume that the current pressure p in the crack is determined by a certain equation of state.

We will introduce the distribution density of the crack f(t, Z, Ω), so that $f dZ d\Omega \times (4\pi)^{-1}$ is the number of cracks per unit volume, the set of parameters of which lies in the limits dZ, dΩ in the region of Z, Ω. Denoting by N the total number of cracks per unit volume (this number may, generally speaking, vary with time) we have

$$\int f(t, Z, \Omega)\, dZ d\Omega = 4\pi N. \tag{17'}$$

With this definition of f we obtain from relations (11) and (12), changing from summation to integration and using (13) and (15),

$$j_k = \lambda_0 \left\{ G_k - \frac{2}{3\pi} \int \frac{s a^3}{s+\gamma} n_k M f dZ d\Omega \right\}. \tag{18}$$

Hence we see that the contribution from each crack only becomes finite when a value of s is reached (much less than unity) on the order of γ or greater. For a crack with a negligibly small initial thickness this begins for smaller loads opening the crack, the higher the electrical resistance of the medium inside it. The effect of all the cracks in this case is of the order of the product of their number per unit volume by the cube of the characteristic radius, which is a small parameter; the integral in (18) is the first term of the expansion with respect to this parameter.

For a system of parallel cracks, e.g., with normals orientated along the first axis, in the given approximation they will only affect j_1. In this case

$$j_1 = \lambda_0 \left\{ G_1 - \frac{8}{3} \int \frac{s a^3}{s+\gamma} M f dZ \right\}, \tag{19}$$

where when calculating M and s from (16) and (17) we have $G_l n_l = G_1$, $\sigma_{lm} n_l n_m = \sigma_{11}$.

If the mechanical field is constant ($s = s_0$), while the electric field undergoes sinusoidal oscillations and is no longer in the steady state, so that G_l and j_l are proportional to $\exp(i\omega t)$, where ω is the angular frequency, then we obtain for the complex amplitudes (we denote them by the same letters) from (16) to (18) assuming to to be fairly large,

$$j_k = \lambda_{kl} G_l, \quad \lambda_{kl} = \lambda_0 \left[\delta_{kl} - \frac{2}{3\pi} \int \frac{s a^3 n_k n_l}{(s+\gamma)(1 - i\omega\tau_s)} f dZ d\Omega \right]. \tag{20}$$

For ω = 0 and γ = 0 Eq. (20) becomes the expression obtained in [3] for the effective resistance of a body with nonconducting cracks. When ω ≠ 0 it can be seen from (20) that there are phase shifts between the components of the current density and the electric field strength, which depend on σ_{kl} and p. One can determine σ_{kl} and p from these shifts.

Consider a simple example. We will assume that the cracks are nonconducting ($\gamma \ll s_*$), perpendicular to the first axis, the pressures in them are the same, and the elastic opening of the cracks is fairly large in the sense mentioned above. Then, from (20), (16), and (17), assuming that this system of cracks only affects the relation between j_1 and G_1, we have

$$j_\perp = \lambda G_1, \quad \lambda = \lambda_0 \left(1 - \frac{8}{3} v \frac{1}{1 - i\omega\tau_s}\right), \quad v = N\overline{a^3}, \tag{21}$$

$$\tau_s = \frac{2\varepsilon_c E_0}{\lambda_0 (1 - \nu_0^2)(\sigma_{11} + p)}, \tag{22}$$

where the bar denotes averaging over the distribution of the radii of the cracks. Hence we find that the current density will lag in phase behind the field strength by a small angle δ, approximately coinciding with its tangent, which, to within terms of higher order of smallness in v, is given by

$$\delta = \frac{8}{3} v \frac{\omega\tau_s}{1 + \omega^2\tau_s^2}. \tag{23}$$

The maximum phase shift occurs at a frequency $\omega_m = \tau_s^{-1}$ and the value of this shift is $\delta_m = {}^4/_3 v$. From these relations, using (22), we can find $\sigma_{11} + p$ (or one of these quantities when the other is known) and $v = (3/4)\delta_m$.

We will now assume that in this system transient cracks occur due to variability of the load, say, a pulsed change with an initial steady state with $s = s_0$, $j_{1_\infty} = j_1(0)$, $G = G_1(0)$, keeping G_1 unchanged. We will assume that $s = s_0 + \chi\delta(t)$, where $\chi = \int_{-\infty}^{\infty} [s(t) - s_0]dt$ and s is given by Eq. (17) with $\sigma_{lm}n_l n_m$ replaced by σ_{11}. As a result, we find from (16) and (19)

$$\frac{j_1 - j_{1(0)}}{j_{1(0)}} = \frac{8}{3} v \left[1 - \exp\left(-\frac{\chi}{\tau}\right)\right] \exp\left(-\frac{s_0 t}{\tau}\right). \tag{24}$$

In semilogarithmic coordinates this relationship can be expressed by a straight line, from the angular coefficient of which we can obtain s_0, and from the intersection on the ordinate axis of which we can obtain v; in the case of a fairly large stretching pulse ($\chi \gg \tau$) v can be found without knowing χ.

For an isotropic system of cracks and an isotropic stressed state ($\sigma_{ik} = \sigma\delta_{ik}$) we can use the same equations (23) and (24) in which 8/3 is replaced by 8/9 and σ_{11} is replaced by σ.

Note that for motions of the elements of the body G is supplemented by a term of the order of the product of their velocity u by the magnetic induction produced by the currents flowing in the body, equal in order of magnitude to $\lambda_0 G a\mu$ [2]. (This term is ignored in this approximation.) Consequently, it is necessary that the condition for this term to be small compared with G be satisfied, i.e.,

$$u \ll (\lambda_0 a\mu)^{-1}. \tag{25}$$

Even under dynamic conditions this condition is satisfied over a wide range of material properties (the electric and magnetic quantities are expressed in a rational system) and crack dimensions.

In order that the consideration of the transient electrical phenomena occurring during a time of the order of τ_s should make sense, condition (25) must be satisfied in all cases for u of the order of $as\tau_s^{-1}$. This, as is easily seen, leads to a condition of the form (10) with a right-hand side $s_*^{-1/2}$ times greater, i.e., this is certainly the case if (10) is satisfied.

NOTATION

a, c_0, and w, radius, initial half-thickness at the center, and the elastic semiopening of the crack; p, pressure in the crack; σ, component of the mechanical stress far from the crack normal to the crack; E_0, Young's modulus; ν_0, Poisson's ratio; λ_0, electrical conductivity of the material; ε_0, constant permittivity or (for a good conducting material) the corresponding index value; ε_c, permittivity of the medium in the crack; λ_c, electrical conductivity of the medium; G and j, electric field and current density vectors, respectively; the index (i) denotes their values inside the crack; and $\perp$ and $\parallel$, components of these vectors perpendicular and parallel to the plane of the crack, respectively; t, time; t_0, time when the process j_k begins; G_k, Cartesian components of the current density and electric field vectors averaged over the volume; σ_{kl}, components of the stress tensor; n_k, components of the vector of the unit normal to the crack; ϑ and ψ, spherical width and length specifying the direction of this vector; $Z = \{a, c_0, t_0, p_0, \varepsilon_c, \lambda_c\}$; p_0, initial value of p; $\Omega = \{\vartheta, \psi\}$, $f(Z, \Omega)$, distribution function; N, number of cracks per unit volume; $\bar{a}^3$, root mean cubic radius of the crack; and ω, angular frequency of the electric field oscillations.

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NUMERICAL SOLUTION OF A NONLINEAR POISSON EQUATION

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A method that is convenient for practical applications is proposed for solving the nonlinear Poisson equation.

A particular class of problems for heat exchange and for magnetohydrodynamics leads to the solution of the Poisson equation with a substantial nonlinearity on the right-hand side.

We consider the Dirichlet problem for Poisson's equation

$$\Delta u(x, y) = f(x, y, u), \quad u|_\Gamma = \varphi(x, y). \tag{1}$$

To simplify the discussion, we assume the region to be rectangular. Using quasilinearization [1] we construct the following iterational process:

$$\Delta v - f_u'(w)\, v = f(w) - f_u'(w)\, w, \tag{2}$$

where $w = u^{(n)}$; $v = u^{(n+1)}$; n is the number of the iteration.

The iterational process (2) ensures quadratic convergence for the condition of exact solution of (2) with fixed right-hand side for each iteration [1].

To determine the values of v for each iteration we use a method of incomplete factorization, similar to that described in [2]. But unlike what was assumed in [2] the splitting of the initial difference operator is represented in the form of the composition of two operators with variable coefficients.

We consider the difference analog of Eq. (2)

$$\Lambda_h v^m = q^m(v) + O(h^3), \tag{3}$$

where m is the index of the iteration for solution of the n-th of Eqs. (2); q^m is the right-hand side of (2) with correction, ensuring the required order of approximation; h equals the maximum of the steps h_x and h_y along the horizontal and vertical directions.

On a nine-point pattern we represent the solution of (3) in the form

$$\begin{gathered} \alpha_{ij} v_{ij}^m + v_{i-1,j}^m + \alpha_{ij} v_{i+1,j}^m = z_{ij}^m; \\ a_{ij} z_{i,j+1}^m + b_{ij} z_{ij}^m + a_{ij} z_{i,j-1}^m = q_{ij}^m, \end{gathered} \tag{4}$$

where the lower indices have the usual meaning. The difference operator on the left side of (3) can easily be transformed by two successive pivots with total number of operations $O(N)$, where N is the number of points of the grid.

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